

PLANES IN DEGENERATE 3-MANIFOLDS

MAHAN MJ

ABSTRACT. We study totally geodesic planes in hyperbolic 3-manifolds M having incompressible core and degenerate ends. We prove a Ratner-type phenomenon: a closed minimal $PSL_2(\mathbb{R})$ -invariant subset of M is either an immersed totally geodesic surface or all of M .

We also show that for an arbitrary infinite volume hyperbolic 3-manifold M without parabolics and with finitely generated fundamental group, the number of compact totally geodesic surfaces in M is finite.

CONTENTS

1. Introduction	1
1.1. Notation and Scheme	2
2. Preliminaries	4
2.1. Cannon-Thurston maps	4
2.2. Algebraic Laminations and Cannon-Thurston Maps	5
2.3. Zariski Dense Subgroups	6
3. H-minimal sets	7
4. Horocycles and geodesics	8
5. Algebraic Laminations	9
5.1. Intersections of \tilde{X} with $\text{bdy}(\tilde{K})$	10
5.2. Quasigeodesics in \tilde{M}	12
6. The Main Theorem	14
6.1. Reduction to two cases	14
6.2. Proof of Main Theorem in Case A	15
6.3. Proof of Main Theorem in Case B	15
6.4. Remarks on the compressible core case	16
7. Compact totally geodesic surfaces	16
7.1. Almost minimizing geodesics	16
7.2. Finitely many closed surfaces	17
Acknowledgments	20
References	20

1. INTRODUCTION

The aim of this paper is to prove the following Ratner-type phenomenon for degenerate hyperbolic 3-manifolds (see Theorem 6.2):

Date: April 8, 2016.

2010 *Mathematics Subject Classification.* 30F40, 37A17, 37B20, 57M50.

Research partially supported by a DST J C Bose Fellowship.

Theorem 1.1. *Let Γ be a degenerate Kleinian group without parabolics and $M = \mathbb{H}^3/\Gamma$ be the associated degenerate hyperbolic 3-manifold. Further, assume that the compact core K of M is incompressible. Let X be a minimal closed H -invariant subset, where $H = PSL_2(\mathbb{R})$. Then X is either an immersed totally geodesic surface or all of M .*

Theorem 1.1 has a long history going back to Hedlund's Theorem [Hed36]. Work of Margulis [Mar89] and Ratner's spectacular resolution of Raghunathan's conjecture [Rat91b, Rat91a] gave a complete answer to the analogous problem for lattices in semi-simple Lie groups. The special case in dimension three [Sha91] says that for a complete hyperbolic 3-manifold of finite volume, any closed H -invariant subset is either an immersed totally geodesic surface or all of the ambient manifold. Theorem 1.1 is thus an exact analog of this Theorem for degenerate hyperbolic 3-manifolds.

The problem of classifying closed K -invariant subsets of a manifold M whose geometry is modeled on a larger (semi-simple) group G makes sense for any pair (G, K) whenever $K \subset G$ and is of interest when K is generated by unipotents [Rat91b]. However the problem has received considerably less attention for infinite volume manifolds. Recently, McMullen, Mohammadi and Oh [MMO15a] (resp. [MMO15b]) satisfactorily completed the study of closed H -invariant subsets of rigid acylindrical hyperbolic 3-manifolds when $H = PSL_2(\mathbb{R})$ (resp. the group of real unipotent matrices). They posed the problem for more general classes of hyperbolic 3-manifolds in [MMO15b]. In this paper we devote ourselves to manifolds at the opposite end of the spectrum, viz. hyperbolic 3-manifolds all whose ends are degenerate. We note that this is the first departure from the convex cocompact world.

In the final Section of the paper, we relax the assumption that M has incompressible core. McMullen, Mohammadi and Oh [MMO15a, Theorem B.1] proved that for a convex cocompact Kleinian group Γ , there can be only finitely many compact totally geodesic surfaces in $M = \mathbb{H}^3/\Gamma$. In this paper, we relax the hypothesis of convex cocompactness and prove the following:

Theorem 1.2. *(see Theorem 7.4) Let Γ be a finitely generated Kleinian group without parabolics and let $M = \mathbb{H}^3/\Gamma$. If M has infinite volume, then there can exist only finitely many compact totally geodesic surfaces in M .*

It is curious that, in the present paper, we do not need the assumption of acylindricity that is essential in [MMO15a]. Degeneracy of the ends seems to force some kind of mixing for a totally geodesic plane. As pointed out in [MMO15a], the Ratner-type phenomenon discussed there and in the present paper fails for certain cylindrical convex cocompact manifolds, e.g. for certain quasi-Fuchsian surface groups that are "nearly Fuchsian". Theorem 1.1 in the special case of a doubly degenerate surface group shows that the Ratner-type phenomenon can be re-instated provided we deform the quasi-Fuchsian surface groups to a degenerate limit. Pitching these two facts against each other seems to suggest an intriguing possibility of a "phase-transition" in quasi-Fuchsian space, where surface groups that are "less degenerate" do not exhibit Ratner-type phenomena, while surface groups that are "more degenerate" do.

1.1. Notation and Scheme.

Definition 1.3. *We say that a hyperbolic 3-manifold M is **degenerate**, if all the ends of M are degenerate. The corresponding Kleinian group Γ will be called a **degenerate Kleinian group**.*

Note that a finitely generated Kleinian group Γ is degenerate if and only if its limit set $\Lambda(= \Lambda_\Gamma)$ is all of S^2 or, equivalently, if the convex hull of Λ is all of \mathbf{H}^3 . Note also that if Γ is abstractly isomorphic to the fundamental group of a closed surface S of genus greater than one, then Γ is degenerate in the sense of Definition 1.3 if and only if Γ is **doubly** or **totally** degenerate in the sense of [Thu80].

We fix, once and for all, the following notation for this paper:

- (1) $G = PSL_2(\mathbb{C})$
- (2) $H = PSL_2(\mathbb{R})$
- (3) A will denote the diagonal matrices in G .
- (4) N will denote the upper triangular unipotent matrices.
- (5) U will denote the *real* upper triangular unipotent matrices.
- (6) V will denote the upper triangular unipotent matrices with purely imaginary off-diagonal entry.
- (7) Throughout most of the paper, Γ will denote a degenerate Kleinian group, except in the last Section, where it will denote an arbitrary finitely generated Kleinian group.
- (8) $M = \mathbb{H}^3/\Gamma$
- (9) K is a compact core of M .
- (10) S^2 will denote the boundary sphere of \mathbb{H}^3
- (11) $\mathcal{S} = PSL_2(\mathbb{C})/PSL_2(\mathbb{R})$ denotes the space of circles on S^2 , equipped with the quotient topology.
- (12) $\mathcal{U} = PSL_2(\mathbb{C})/U$ denotes the space of horocycles in \mathbb{H}^3 , equipped with the quotient topology.
- (13) X will be a minimal closed H -invariant subset of M .
- (14) Y will be a minimal closed U -invariant subset of M .
- (15) $B^3 = \mathbf{H} \cup S^2$ is the usual compactification of \mathbb{H}^3 .

We are interested in studying closed H -invariant subsets of M . This is equivalent to studying closed Γ -invariant subsets of \mathcal{S} (see [MMO15a] for more on this correspondence). We elaborate here on the topology on \mathcal{S} . Equip the collection of closed subsets of S^2 with the Hausdorff topology and let $C_c(S^2)$ denote the subspace consisting of subsets that have cardinality greater than one, i.e. we exclude singletons. Then \mathcal{S} is contained in $C_c(S^2)$ and the topology coincides with the subspace topology. Theorem 1.1 then asserts that a minimal closed Γ -invariant subset of \mathcal{S} with the subspace topology inherited from $C_c(S^2)$ is either discrete or all of \mathcal{S} .

A recurring theme (cf. [Mar89]) in the study of closed H -invariant sets X is that it leads us naturally to the study of U -invariant sets Y , where U is the set of real unipotent matrices. U -orbits coincide with horocycles. A word about the topology on \mathcal{U} . Adjoining the basepoint of a horocycle to the horocycle we obtain a circle in B^3 . Equip the collection of closed subsets of B^3 with the Hausdorff topology and let $C_c(B^3)$ denote the subspace consisting of subsets that have cardinality greater than one, i.e. we exclude singletons. Then \mathcal{U} is contained in $C_c(B^3)$ and the topology coincides with the subspace topology.

We are now in a position to discuss the scheme of the paper. Section 2 recalls the two main ingredients that feed into the paper:

- (1) The existence and structure of Cannon-Thurston maps [Mj14a, Mj14b, DM16, Mj10] for arbitrary finitely generated Kleinian groups.
- (2) Some general structure theorems from [MMO15a].

Section 3 establishes the existence of minimal closed H -invariant sets X . Our strategy for the next three sections is to construct within X , sufficiently complicated minimal closed U -invariant sets Y . Here ‘sufficiently complicated’ simply means that Y contains more than one horocycle. The construction of Y is indirect. Section 4 establishes a general correspondence between geodesic rays that recur to a compact subset of M and recurrent horocycles. This reduces the problem of constructing Y to the construction of recurrent geodesic rays. Section 5 then shows that (roughly speaking) the intersection of X with a suitably chosen compact core K of M furnishes an algebraic lamination (traditionally the support of a geodesic current) \mathcal{C} . Using the structure theory of Cannon-Thurston maps, it then suffices to show that \mathcal{C} contains no ending lamination (cf. [Thu80, Min10, BCM12]). This, last, is accomplished by a geometric limit argument.

With the existence of closed U -minimal sets Y in place, the proof of Theorem 1.1, carried out in Section 6, is largely similar to the corresponding argument in [MMO15a]. In Section 7, we extend a Theorem of McMullen, Mohammadi and Oh [MMO15a, Theorem B.1]: we show that in any infinite volume hyperbolic 3-manifold with finitely generated fundamental group, there can be only finitely many compact totally geodesic immersed surfaces.

A final word about the difference in techniques between [MMO15a] and the present paper. The former uses thick sets, the renormalized frame flow and mixing properties of the frame flow in an essential way, as the focus there is on ‘reducing’ the problem (in a sense) to the convex core, which in turn has properties resembling a compact manifold. The convex core is, in fact, a compact manifold with boundary. Our focus in this paper, is topological-dynamic in flavor and a detailed analysis of the Γ -action on its limit set (as explicated by the Cannon-Thurston map) is the main ingredient. By their very nature, the examples dealt with in this paper permit no compact “reduction”. In this sense, the examples of hyperbolic 3-manifolds explored here are the first genuinely infinite volume examples for which a Ratner-type phenomenon is obtained.

2. PRELIMINARIES

2.1. Cannon-Thurston maps.

Definition 2.1. *Let W and Z be hyperbolic metric spaces and $i : Z \rightarrow W$ be an embedding. Suppose that a continuous extension $\hat{i} : \widehat{Z} \rightarrow \widehat{W}$ of i exists between their (Gromov) compactifications. Then the boundary value of \hat{i} , namely $\partial i : \partial Z \rightarrow \partial W$ is called a **Cannon-Thurston map**.*

For us, Z will be a Cayley graph of the Kleinian group Γ with respect to a finite generating set and W will be \mathbf{H}^3 , where we identify (the vertex set of) Z with an orbit of Γ in \mathbf{H}^3 . Equivalently (as is often done in geometric group theory), we may choose a compact core K of M and W, Z will be identified with \widetilde{M} and \widetilde{K} respectively. Then the main Theorems of [Mj14a, DM16, Mj14b, Mj10] give us:

Theorem 2.2. *Let M be a degenerate hyperbolic 3-manifold without parabolics and K a compact core. Let Γ be the corresponding Kleinian group. A Cannon-Thurston map ∂i exists for $i : \Gamma \rightarrow \widetilde{M}$ (or equivalently, $i : \widetilde{K} \rightarrow \widetilde{M}$). Further, ∂i identifies $a, b \in \partial\Gamma$ iff a, b are end-points of a leaf of an ending lamination \mathcal{L}_E or boundary points of an ideal polygon whose sides are leaves of \mathcal{L}_E , where E is an end of M .*

In Section 4, we shall be concerned with the conical limit set of a Kleinian group. We furnish some definitions here and explicate the relationship with Cannon-Thurston maps.

Definition 2.3. *A point $z \in \Lambda (= \Lambda_\Gamma)$ is called a conical limit point if for any base-point o , there exist*

- (1) *a geodesic ray γ landing at z*
- (2) *$R > 0$*

such that the neighborhood $N_R(\gamma)$ contains infinitely many distinct orbit points $g.o$, $g \in \Gamma$.

The collection $\Lambda_c (= \Lambda_c(\Gamma))$ of conical limit points is called the conical limit set of Γ .

Remark 2.4. The Cannon-Thurston map in Theorem 2.2 furnishes a method of pulling back any subset of S^2 to $\partial\widetilde{K}$. For any totally geodesic plane $\widetilde{P} \subset \mathbb{H}^3$, $(\partial i)^{-1}(\partial\widetilde{P})$ is a closed subset of $\partial\widetilde{K}$, the Gromov boundary of \widetilde{K} . We denote $(\partial i)^{-1}(\partial\widetilde{P})$ by \widetilde{P}_K for short. Again, $(\partial i)^{-1}(\Lambda_c)$ is a subset of $\partial\widetilde{K}$, which we denote as $\Lambda_{c,K}$. Section 5 can be viewed as an investigation into the properties of $\widetilde{P}_K \cap \Lambda_{c,K}$; in particular we establish that this intersection is non-empty.

A heuristic reason to believe that the intersection is non-empty for doubly degenerate surface Kleinian groups is the following:

A doubly degenerate M has two ends, the $+\infty$ -end and the $-\infty$ -end say. Geodesics in M that do not correspond to points in Λ_c necessarily have to exit M through either the $+\infty$ -end or the $-\infty$ -end. It is therefore natural to expect that $\partial\widetilde{P}$ cannot be built up entirely of these two types of points and anything "in between" necessarily give conical limit points.

Lemma 2.5. *Let P be a totally geodesic immersed plane in M and \widetilde{P} be a lift to \widetilde{M} . Let E be an incompressible end of M and $S = \partial E$ be its boundary surface. Let $\partial i : \partial\widetilde{S} \rightarrow \partial\widetilde{M}$ denote the Cannon-Thurston map for $S \hookrightarrow M$. Then $\widetilde{P}_S := (\partial i)^{-1}(\partial\widetilde{P})$ is homeomorphic to a Cantor set contained in the circle $\partial\widetilde{S}$.*

Proof. First, since ∂i is continuous \widetilde{P}_S is closed. Further, since S^1 is perfect and ∂i is finite-to-one, \widetilde{P}_S is perfect. Also, for any interval $I \subset \partial\widetilde{S}$, ∂i identifies pairs of points corresponding precisely to end-points of leaves of \mathcal{L}_E , the ending lamination corresponding to E (by Theorem 2.2). In particular, I cannot map to a subarc of $\partial\widetilde{P}$. In other words \widetilde{P}_S has empty interior. \square

2.2. Algebraic Laminations and Cannon-Thurston Maps.

Definition 2.6. *An algebraic lamination [BFH97, Mit97, CHL08, KL10] for a hyperbolic group Γ is a Γ -invariant, closed subset $\mathcal{L} \subseteq \partial^2\Gamma = (\partial\Gamma \times \partial\Gamma \setminus \Delta) / \sim$, where $(x, y) \sim (y, x)$ denotes the flip and Δ is the diagonal in $\partial\Gamma \times \partial\Gamma$.*

If $(p, q) \in \mathcal{L}$, then any bi-infinite geodesic joining p, q in Γ is called a leaf of \mathcal{L} .

Cannon-Thurston maps, when they exist automatically define a natural algebraic lamination.

Definition 2.7. [Mit99] *Suppose that a Cannon-Thurston map ∂i exists for the pair (Γ, W) . We define the Cannon-Thurston lamination \mathcal{L}_{CT} as follows: $\mathcal{L}_{CT} = \{(p, q) \in \partial^2 \Gamma \mid p \neq q \text{ and } \partial i(p) = \partial i(q)\}$.*

The relationship between algebraic laminations, Cannon-Thurston maps and quasiconvexity has been explored recently in [MR15]. We note the following facts that will be relevant for us.

Lemma 2.8. [Mit99, Lemma 2.1] *Let Γ be a hyperbolic group acting on a hyperbolic metric space W such that a Cannon-Thurston map exists for the pair (Γ, W) . Then some (any) Γ -orbit is quasiconvex if and only if $\mathcal{L}_{CT} = \emptyset$.*

Lemma 2.9. *Let Γ be a hyperbolic group acting on a hyperbolic metric space W such that a Cannon-Thurston map exists for the map $i : \Gamma \rightarrow W$ identifying Γ with its orbit in W . Let \mathcal{L} be an algebraic lamination in $\partial^2 \Gamma$. Then exactly one of the following holds:*

- (1) *Either there exists C_0 such that for every leaf l of \mathcal{L} , $i(l)$ is a C_0 -quasigeodesic in W .*
- (2) *Or $\mathcal{L} \cap \mathcal{L}_{CT} \neq \emptyset$ and hence contains a minimal closed Γ -invariant subset of \mathcal{L}_{CT} . Further, Γ orbits are not quasiconvex.*

Proof. We identify Γ with a Cayley graph (with respect to a finite generating set).

Suppose that Alternative (1) fails. Then for all $n \geq 0$, there exist leaves l_n of \mathcal{L} and $a_n, b_n \in l_n$, $c_n \in [a_n, b_n]$ (the geodesic in Γ joining a_n, b_n) such that the geodesic α_n in W joining a_n, b_n lies outside $N_n(c_n)$, where $N_n(c_n)$ denotes the n -ball around c_n in W . Translating by a group element g_n (in Γ) if necessary, we may assume that c_n is the identity element of Γ .

Since α_n lies outside $N_n(c_n)$, the visual diameter of α_n tends to zero as $n \rightarrow \infty$. Hence, passing to a limit we obtain

- (1) a leaf l_∞ of \mathcal{L} as a limit of the l_n 's (using the fact that \mathcal{L} is closed).
- (2) The Cannon-Thurston map for $i : \Gamma \rightarrow W$ identifies the end-points at infinity of l_∞ .

Hence $l_\infty \subset \mathcal{L} \cap \mathcal{L}_{CT}$. In particular, \mathcal{L}_{CT} is non-empty and hence by Lemma 2.8, Γ orbits are not quasiconvex. Since \mathcal{L}_{CT} is also an algebraic lamination, $\mathcal{L} \cap \mathcal{L}_{CT}$ contains the minimal closed Γ -invariant subset of $\partial^2 \Gamma$ containing l_∞ . Alternative (2) follows. \square

2.3. Zariski Dense Subgroups. The following two Theorems are specializations to our context of general facts from [MMO15a]. The corresponding Theorems in [MMO15a] are established in the general context of Zariski dense subgroups of $PSL_2(\mathbb{C})$. This hypothesis is always satisfied for Zariski dense subgroups.

Theorem 2.10. [MMO15a, Theorem 4.1, Corollary 4.2] *Let Γ be a degenerate Kleinian group and let $\Lambda (= S^2)$ be its limit set. Let $\mathcal{D} \subset \mathcal{S}$ be a collection of circles such that $\bigcup_{D \in \mathcal{D}} D$ contains a nonempty open subset of Λ . Then there exists a $D \in \mathcal{D}$ such that $\overline{\Gamma D} = \mathcal{S}$.*

Theorem 2.11. [MMO15a, Theorem 5.1] *Let Γ be a degenerate Kleinian group. Let $C \subset S^2$ be a circle, and suppose $D \in \overline{\Gamma C} \setminus \Gamma C$. Let $\text{Stab}(D)$ denote the stabilizer of D in Γ . If $\text{Stab}(D)$ is a non-elementary group, then $\overline{\Gamma C} = \mathcal{S}$.*

The next statement is valid for arbitrary hyperbolic 3-manifolds.

Theorem 2.12. [MMO15a, Corollary B.2] *Let N be an arbitrary hyperbolic 3-manifold. Let $K \subset N$ be a compact subset. Then either the set of compact geodesic surfaces in N contained in K is finite; or the union of compact geodesic surfaces is dense in N , and N is compact.*

We shall also need the following general Lie groups fact proved in [MMO15a]:

Theorem 2.13. [MMO15a, Theorem 8.2] *Let $g_n \rightarrow id$ in $G \setminus AN$ and G_0 be a neighborhood of the identity. Then there exist $u_n, u'_n \rightarrow \infty$ in U such that after passing to a subsequence, we have $u_n g_n u'_n \rightarrow g \in G_0 \cap (A \text{ Vid})$.*

3. H-MINIMAL SETS

Lemma 3.1. *Let M be a degenerate manifold and K a compact core of M . Then any totally geodesic immersed plane in M intersects K .*

Proof. Let $f : \mathbb{H}^2 \rightarrow M$ be a totally geodesic immersion and $\tilde{f} : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ be a lift to the universal cover.

Let K be the compact core and $i : K \rightarrow M$ denote inclusion. Then i lifts to $\tilde{i} : \tilde{K} \rightarrow \tilde{M}$.

If $f(P)$ misses $i(K)$, then there exists a lift $\tilde{f}(\mathbb{H}^2)$ missing $\tilde{i}(\tilde{K})$, and so the limit set of $\tilde{i}(\tilde{K})$ lies entirely to one side of the boundary circle $C = \partial(\tilde{f}(\mathbb{H}^2))$. But then the limit set of \tilde{K} is not all of S^2 , contradicting the fact that K is the compact core of a degenerate manifold M . \square

Since a (not necessarily totally geodesic plane) lying in a bounded neighborhood of a totally geodesic plane necessarily has the same limit set, we have the following immediate Corollary:

Corollary 3.2. *Let M be a degenerate manifold and K a compact core of M . Let P be a totally geodesic immersed plane in M , \tilde{P} be a lift to \tilde{M} , \tilde{P}_1 be a plane lying in a bounded neighborhood of \tilde{P} and P_1 be the image of \tilde{P}_1 in M under the covering projection. Then P_1 intersects K .*

Proposition 3.3. *Let M be a degenerate manifold. Let $X \subset M$ be a closed subset that is H -invariant. Equivalently, X is a closed subset of M that is a union of immersed totally geodesic planes in M . Then there exists a minimal closed H -invariant subset $X_0 \subset X$.*

Proof. The proof is standard (cf. [Mar89, MMO15a]) given Lemma 3.1. Let K denote a compact core of M . Any closed H -invariant X_α intersects K by Lemma 3.1. We consider the collection of closed H -invariant subsets $X_\alpha \subset X$ partially ordered as follows: $X_\alpha < X_\beta$ if $X_\alpha \cap K \subset X_\beta \cap K$. Hence, for any totally ordered collection $\{X_\alpha\}$, $\cap_\alpha X_\alpha \cap K$ is non-empty, as K is compact. This forces $\cap_\alpha X_\alpha$ to be non-empty. The existence of a minimal closed invariant subset $X_0 \subset X$ now follows by Zorn's Lemma. \square

A minimal closed H -invariant subset will be called H -**minimal**. We have thus proved the existence of H -minimal sets. We assume henceforth (using Proposition 3.3) that $X = X_0$ is H -minimal. Recall that U denotes the *real* upper triangular unipotent matrices. The next few sections of the paper are devoted to finding sufficiently complicated U -minimal sets inside the H -minimal set X . This will involve a number of ingredients.

4. HOROCYCLES AND GEODESICS

In this section, we relate properly embedded geodesics in M with properly embedded horocycles in M . We note here that for the purposes of this section, we assume only that M is degenerate without parabolics; no hypothesis on incompressibility of its compact core K are imposed. Properly embedded geodesics in M are said to be **exiting**. A geodesic ray in M is exiting if and only if any lift to the universal cover converges to a point $p \in \partial\widetilde{M}$ not belonging to the conical limit set (cf. [LM16, Section 3]). It follows that if γ is a geodesic ray in M such that some (and hence every) lift γ_1 of γ to \widetilde{M} lands on a conical limit point, then there exist

- (1) a compact subset $Q \subset M$
- (2) a sequence of times $t_n \rightarrow \infty$ such that $\gamma(t_n) \in Q$.

In fact non-exiting geodesics are *characterized* by the fact that they recur to a compact subset of M infinitely often. In other words, geodesics in M that recur infinitely often to a compact subset of M correspond precisely to geodesics in \widetilde{M} that land on the conical limit set. We refer the reader to [LM16, Section 3] for a detailed discussion. In Proposition 4.1 we shall show that proper embeddedness of a horocycle forces the corresponding geodesic ray to be exiting. In subsequent sections of this paper, we are going to be interested in horocycles that are **not** properly embedded. Proposition 4.1 then tells us that it suffices to look at geodesics that are not exiting.

Recall that K is the compact core of M . Let X be H -minimal. Let $P \subset X$ be one of the totally geodesic planes comprising X . By minimality the closure $\overline{P} = X$. Let $\sigma \subset P$ be a (bi-infinite) horocycle contained in P passing through the compact core K . Choose $o \in K \cap \sigma$ such that o does not lie in the self-intersection locus of P . It follows that after lifting to the universal cover \widetilde{M} and fixing a lift o_1 of o , there is a unique lift of P through o_1 . Call this P_1 . Let σ_1 be the lift of σ through o_1 . Then $\sigma_1 \subset P_1$. Let τ_1 be the geodesic ray in P_1 starting at o_1 and asymptotic to $\sigma(+\infty) = \sigma(-\infty) \in \partial\widetilde{M}$. Project τ_1 back to M to obtain the geodesic ray τ through o , contained in the (immersed) horodisk in P bounded by σ . We say that τ is the unique geodesic ray through o in P that is **asymptotic** to σ .

Proposition 4.1. *Let σ, P, o be as above and let τ be the unique geodesic ray through o in P asymptotic to σ . If σ is properly embedded in M , then so is τ .*

Proof. As in the discussion before the Proposition, let $o_1, \sigma_1, \tau_1, P_1$ be lifts of o, σ, τ and P respectively. Let D_1 be the horodisk with σ_1 as its boundary and D denote its projection to M .

If $a_n \rightarrow \infty$ and $b_n \rightarrow -\infty$ along σ , the hypothesis guarantees that a_n, b_n exit the same degenerate end E of M . Let (a_n, b_n) denote the subarc of σ joining a_n, b_n and let $[a_n, b_n]$ be the geodesic in D joining a_n, b_n .

We first show that $[a_n, b_n]$ exits E . If not, then there is a compact $Q \subset M$ such that $[a_n, b_n] \cap Q \neq \emptyset$ for all n . Passing to a subsequential limit we obtain a geodesic γ passing through Q and hence (after lifting to \widetilde{M}) $\gamma(\infty) \neq \gamma(-\infty)$, where $\gamma(\infty) = \lim_n a_n$ and $\gamma(-\infty) = \lim_n b_n$ in \widetilde{M} . On the other hand any subsequential limit of (a_n, b_n) is σ (since $a_n \rightarrow \infty$ and $b_n \rightarrow -\infty$ along σ) and hence (after lifting to \widetilde{M}) the limits of a_n and b_n must coincide. This contradiction proves that $[a_n, b_n]$ exits E for any $a_n \rightarrow \infty$ and $b_n \rightarrow -\infty$ along σ . Now, choose $a_n = n$ and $b_n = -n$ and let $t_n = \tau \cap [a_n, b_n]$. Since $[a_n, b_n]$ exits E , it follows that t_n exits E . It follows

that τ exits E (since the distance between t_n and t_{n-1} is uniformly bounded and hence the geodesic joining them in \widetilde{M} is uniformly bounded in length). \square

5. ALGEBRAIC LAMINATIONS

Our aim (till Proposition 6.1 below) is to establish the existence of a sufficiently complicated U -minimal set Y contained in the H -minimal set X constructed in Proposition 3.3. Here "sufficiently complicated" simply means that Y does not consist of a single horocycle.

Our approach in constructing such a Y is indirect. We shall be interested in horocycles that are not properly embedded in M . Proposition 4.1 allows us to turn our attention instead at geodesics that are not exiting. A natural class of non-exiting geodesics are given by those that lie in a compact subset of M . The purpose of this section is to construct such a class of geodesics. Let K be a compact core of M as usual. Roughly speaking, the intersection $X \cap K$ furnishes for us such a family of geodesics.

Let \widetilde{X} , \widetilde{K} and \widetilde{P} be pre-images of X , K and P respectively in \widetilde{M} . If some $P(\subset X)$ is contained in a compact subset Q of M , then by minimality of X , so is X . It follows that all horocycles contained in planes comprising X are contained in Q , and hence none are properly embedded in M . In this situation, we define Y_0 to be the U -invariant subset of G/U given by **all** horocycles contained in planes comprising X . Proposition 6.1 will establish the existence of a U -minimal set Y contained in Y_0 in this situation without much further work (the proof is a reprise of that of Proposition 3.3).

In light of this we work under the following hypothesis for the purposes of this section:

Hypothesis 5.1. *The H -minimal set X furnished by Proposition 3.3 is not contained in any compact subset of M .*

Remark 5.2. *Theorem 2.12 by McMullen, Mohammadi and Oh (or an extension of [Rat91b]) guarantees that Hypothesis 5.1 holds for degenerate M : Either M is compact or X is a closed immersed surface.*

However, we do not need to apply Theorem 2.12 in order to prove Theorem 6.2 below; hence the status of a Hypothesis.

Notation: For K a compact manifold with boundary, we shall denote the boundary by $\text{bdy}(K)$. Thus, for K a compact core of M , $\text{bdy}(\widetilde{K})$ will denote the boundary of \widetilde{K} thought of as a manifold with boundary. We shall reserve the notation $\partial\widetilde{K}$ to denote the **Gromov**-boundary of \widetilde{K} , when the latter is hyperbolic.

As usual, let K be a compact core of M . Consider a lift \widetilde{P} of P to \widetilde{M} . For any component K_0 of $\widetilde{P} \cap \widetilde{K}$, we define the **depth** $\text{inj}(K_0)$ of K_0 to be $\sup_{x \in K_0} d(x, \text{bdy}(\widetilde{K}))$. Thus, the depth of K_0 equals the radius of the largest totally geodesic hyperbolic disk that can be embedded in K_0 . This makes sense as K_0 is a closed subset of the totally geodesic plane \widetilde{P} in \widetilde{M} .

Lemma 5.3. No deep components: *Fix an H -minimal X in M and assume that Hypothesis 5.1 holds. Then for any compact core K of M , there exists R_0 such that for any P in X and any component K_0 of $\widetilde{P} \cap \widetilde{K}$, $\text{inj}(K_0) \leq R_0$.*

Proof. Suppose not. Then for all $n \in \mathbb{N}$, there exists a plane P_n , a lift \tilde{P}_n , a component K_0 of $\tilde{P}_n \cap \tilde{K}$, and $z_n \in K_0$ such that a totally geodesic disk of radius n about z_n is contained in K_0 . Translating by an element of $\Gamma (= \pi_1(M) = \pi_1(K))$, we may assume that z_n lies inside a fundamental domain for the Γ -action on \tilde{K} . Passing to a limit, as $n \rightarrow \infty$, we obtain a totally geodesic (infinite) plane P_∞ contained inside K . Since X is H -minimal, it equals the closure of P_∞ and hence $X \subset K$, contradicting Hypothesis 5.1. \square

5.1. Intersections of \tilde{X} with $\text{bdy}(\tilde{K})$. In this subsection, we give a topological argument to show that ‘spurious’ intersections of \tilde{X} with $\text{bdy}(\tilde{K})$ can be removed by standard topological surgeries.

Removing Inessential Loops: Consider a lift \tilde{P} of P to \tilde{M} and assume (after perturbing K slightly in M) that \tilde{P} is transverse to $\text{bdy}(\tilde{K})$. A compact connected component σ (necessarily homeomorphic to S^1) of $\tilde{P} \cap \text{bdy}(\tilde{K})$ is called an **inessential loop** if σ is homotopically trivial in $\text{bdy}(\tilde{K})$. Equivalently, σ bounds a disk in the boundary $\text{bdy}(\tilde{K})$. We may replace every bounded component K_0 of $\tilde{P} \cap \tilde{K}$ by a subsurface K_1 of $\tilde{P} \cap \text{bdy}(\tilde{K})$ having the same boundary circles. Since there are no deep components by Lemma 5.3 (under Hypothesis 5.1), $\text{inj}(K_0) \leq R_0$ for all K_0 . Since $\text{bdy}(\tilde{K})$ is uniformly properly embedded in \tilde{M} , it follows immediately that the injectivity radius of K_1 is bounded in terms of R_0 . Hence replacing each such K_0 by the corresponding K_1 , we obtain a surface \tilde{P}_1 lying in a bounded neighborhood of \tilde{P} . Also they have the same boundary circle $\partial\tilde{P}_1 = \partial\tilde{P}$. Replacing \tilde{P} by \tilde{P}_1 (not necessarily totally geodesic), and taking the minimal closed Γ -invariant set generated by \tilde{P}_1 , we obtain a new minimal set X_1 in M having no inessential loops in $\tilde{K} \cap \tilde{X}_1$.

Note that though each \tilde{X}_1 is no longer H -invariant, the collection $\{\partial\tilde{P}_1 : P_1 \in X_1\}$ does agree with $\Gamma\partial\tilde{P} \subset \mathcal{S}$. Since \tilde{X}_1 will serve only an auxiliary purpose in what follows, this is not going to cause a problem in what follows.

Removing Asymptotically Inessential Loops: Let $P, \tilde{P}, K, \tilde{K}$ be as above; in particular \tilde{P} is transverse to $\text{bdy}(\tilde{K})$. A bi-infinite path σ in $\tilde{P} \cap \text{bdy}(\tilde{K})$ is called a C_0 -**asymptotically inessential loop** if it is not properly embedded, or equivalently in our situation, there exist $a_n \rightarrow \infty, b_n \rightarrow -\infty$ along σ such that $d(a_n, b_n)$ is bounded by C_0 .

Such a bi-infinite path σ gives “almost closed loops” in the following sense. There exists arcs τ_n (resp. θ_n) of length bounded in terms of C_0 in $\text{bdy}(\tilde{K})$ (resp. \tilde{P}) such that

- (1) τ_n (resp. θ_n) along with arbitrarily long subarcs σ_n joining a_n, b_n bound contractible loops α_n (resp. β_n) in $\text{bdy}(\tilde{K})$ (resp. \tilde{P}). The loops α_n (resp. β_n) bound disks Δ_{1n} (resp. Δ_{2n})
- (2) the loops $\tau_n \cup \theta_n$ bound disks Δ_{3n} lying in a bounded neighborhood of $\text{bdy}(\tilde{K})$.

Again by homotoping \tilde{P} by a bounded amount D_0 (depending only on C_0), one can get rid of C_0 -asymptotically inessential loops. This is done iteratively over n :

replace Δ_{2n} by $\Delta_{1n} \cup \Delta_{3n}$ and let $n \rightarrow \infty$.

Two Alternatives: We are now in a position to describe intersections of \tilde{X} with $\text{bdy}(\tilde{K})$. We shall note below that we can reduce \tilde{X} with $\text{bdy}(\tilde{K})$ to essentially two kinds of intersections that survive removal of inessential loops and asymptotically inessential loops:

- Alternative A: A (closed $\pi_1(M)$ –invariant) collection \mathcal{C} of bi-infinite paths in $\tilde{X} \cap \text{bdy}(\tilde{K})$ properly embedded in $\text{bdy}(\tilde{K})$.
 Alternative B: A (closed $\pi_1(M)$ –invariant) collection \mathcal{D} of compressing disks in $\tilde{X} \cap \text{bdy}(\tilde{K})$ where each $D \in \mathcal{D}$ has diameter bounded by some d_0 .

We describe now how the two above alternatives are obtained. First, by homotoping K and X , we can remove inessential loops. Next, for any fixed C_0 , we can get rid of C_0 –asymptotically inessential loops by a similar homotopy. Two cases arise now. If all asymptotically inessential loops are C_0 –asymptotically inessential for some $C_0 > 0$, then we can remove them all. Else, $\tilde{X} \cap \text{bdy}(\tilde{K})$ contains (Gromov-Hausdorff) limits of n –asymptotically inessential loops, as n tends to infinity. Such limits are necessarily bi-infinite paths in $\tilde{X} \cap \text{bdy}(\tilde{K})$ properly embedded in $\text{bdy}(\tilde{K})$. In short, either we can remove all asymptotically inessential loops or Alternative A holds.

Suppose therefore that Alternative A fails to hold and also (by homotopy) inessential loops as well as asymptotically inessential loops have been removed. We shall show that in this situation, Alternative 2 holds. Since \mathcal{C} is empty, all intersections come from simple closed curves in $\text{bdy}(\tilde{K})$ that are compressible in M , and hence in K . The innermost curves necessarily correspond to compressing disks in $\tilde{X} \cap \text{bdy}(\tilde{K})$. If there exists a sequence D_i in this collection, such that $\text{dia}(D_i) \rightarrow \infty$, then, by Lemma 5.3, they have uniformly bounded depth and their boundary circles necessarily limit to paths in $\text{bdy}(\tilde{K})$ satisfying Alternative A, contradicting our assumption. Hence Alternative 2 holds. We summarize this discussion as follows:

Proposition 5.4. *Let M be a degenerate hyperbolic 3-manifold with K a compact core. Let X be an H –minimal set in M not contained in a compact subset. Then $\tilde{X} \cap \text{bdy}(\tilde{K})$ satisfies at least one of the following alternatives:*

- Alternative A: $\tilde{X} \cap \text{bdy}(\tilde{K})$ contains a (closed $\pi_1(M)$ –invariant) collection \mathcal{C} of bi-infinite paths in $\tilde{X} \cap \text{bdy}(\tilde{K})$ properly embedded in $\text{bdy}(\tilde{K})$.
 Alternative B: $\tilde{X} \cap \text{bdy}(\tilde{K})$ contains a (closed $\pi_1(M)$ –invariant) collection \mathcal{D} of compressing disks in $\tilde{X} \cap \text{bdy}(\tilde{K})$ where each $D \in \mathcal{D}$ has diameter bounded by some d_0 .

As an immediate consequence, we have:

Lemma 5.5. *Suppose that M has an incompressible core, i.e. $\text{bdy}(K)$ is incompressible in M . Then, $\tilde{X} \cap \text{bdy}(\tilde{K})$ contains a (closed $\pi_1(M)$ –invariant) collection \mathcal{C} of bi-infinite paths in $\tilde{X} \cap \text{bdy}(\tilde{K})$ properly embedded in $\text{bdy}(\tilde{K})$.*

We now proceed to deal with Alternative A. Each bi-infinite path l in \mathcal{C} lifted to $\text{bdy}(\tilde{K})$ will be called a **leaf** of \mathcal{C} .

Lemma 5.6. *There exists C such that for any leaf l of \mathcal{C} lifted to $\text{bdy}(\tilde{K})$ and any $a, b \in l$, $l \subset N_C([a, b])$, where $N_C([a, b])$ denotes the C -neighborhood of the geodesic $[a, b]$ joining a, b in $\text{bdy}(\tilde{K})$.*

Proof. Suppose not. Then for every positive integer n , there exist leaves l_n and $a_n, b_n \in l_n$, $c_n \in [a_n, b_n]$ (the geodesic in \tilde{K} joining a_n, b_n) such that $l \cap N_n(c_n) = \emptyset$, where $N_n(c_n)$ denotes the n -ball around c_n in \tilde{K} . Since l_n is a (necessarily connected) path in $\text{bdy}(\tilde{K})$, we may assume that there exists a component \tilde{S} of $\text{bdy}(\tilde{K})$, such that $[a_n, b_n]$ tracks a quasigeodesic in \tilde{S} . Without loss of generality therefore, we assume (after passing to a subsequence if necessary) that $[a_n, b_n]$ is contained in \tilde{S} for all n .

Translating by a group element g_n (in $\pi_1(K)$) if necessary, we may assume that c_n lies in a fixed fundamental domain in \tilde{K} . Passing to a subsequence we have a sequence of paths α_n in \tilde{S} joining a_n, b_n and lying outside an $(n - D_0)$ -ball about a fixed base point, where D_0 is the diameter of a fundamental domain in \tilde{K} . Let $P_n \subset \tilde{M}$ be the sequence of totally geodesic planes containing α_n . Passing to a subsequence we may assume that the sequence $\{P_n \cup \partial(P_n)\}$ converges in the Hausdorff topology on compact subsets of $\tilde{M} \cup \partial\tilde{M} = B^3$. Since each $\{P_n \cup \partial(P_n)\}$ is a round disk (say in the projective model), the limit $P_\infty \cup \partial P_\infty$ is necessarily a round disk passing through a fixed fundamental domain in \tilde{K} .

In particular, the preimage under the Cannon-Thurston map ∂i of ∂P_∞ is a Cantor set in the boundary $\partial\tilde{S}$ by Lemma 2.5. On the other hand $[a_n, b_n]$ converges to a bi-infinite geodesic (a_∞, b_∞) and hence α_n converges to an arc in $\partial\tilde{S}$ joining a_∞, b_∞ in $\partial\tilde{S}$. This contradiction yields the Lemma. \square

As an immediate Corollary we have

Corollary 5.7. *There exists C such that any leaf l of \mathcal{C} in \tilde{K} is a C -quasigeodesic in \tilde{K} .*

Remark 5.8. We shall say that a leaf l is **carried by** a component $\tilde{S}(\subset \text{bdy}(\tilde{K}))$ if it has a quasigeodesic representative contained in \tilde{S} . In this situation we shall identify l with such a quasigeodesic representative. By Corollary 5.7, we may assume without loss of generality that \mathcal{C} is an algebraic lamination, i.e. a closed $\pi_1(K)$ -invariant collection of bi-infinite geodesics in $\text{bdy}(\tilde{K})$, or equivalently, a closed $\pi_1(K)$ -invariant subset of $\partial^2\Gamma$. Further, the proof of Lemma 5.6 shows that there exists a component S of $\text{bdy}(K)$, such that its preimage in \tilde{K} carries a closed $\pi_1(K)$ -invariant subset of \mathcal{C} (when Alternative A holds).

5.2. Quasigeodesics in \tilde{M} . We have already shown that leaves of \mathcal{C} are uniform quasigeodesics in \tilde{K} . In this subsection we show that, moreover, leaves of \mathcal{C} are uniform quasigeodesics in \tilde{M} (when Alternative A holds).

Lemma 5.9. *Let M be a degenerate hyperbolic manifold and let S be a boundary component of a compact core K of M . Let \mathcal{C}_0 be an algebraic lamination in $\partial^2\Gamma$, whose leaves are carried by \tilde{S} . Then exactly one of the following holds:*

- (1) *Either there exists C_0 such that every leaf of \mathcal{C}_0 is a C_0 -quasigeodesic in \tilde{M} .*
- (2) *Or there exists an end E such that $\mathcal{L}_E \subset \mathcal{C}_0$, where \mathcal{L}_E is the ending lamination for an end E with $\partial E = S$.*

Proof. Suppose that alternative (1) does not hold. Then, by Lemma 2.9, $\mathcal{C}_0 \cap \mathcal{L}_{CT} \neq \emptyset$, where \mathcal{L}_{CT} denotes the Cannon-Thurston lamination for the action of Γ on \mathbb{H}^3 .

By Theorem 2.2, there exists a boundary component S of the compact core K such that l_∞ is either a leaf of an ending lamination \mathcal{L}_E for some end E with boundary S or a diagonal of a complementary ideal polygon. Since the smallest closed $\pi_1(S)$ -invariant subset of $(S^1 \times S^1 \setminus \Delta)/\sim$ containing such an l_∞ is all of \mathcal{L}_E , it follows that alternative (2) holds. \square

Lemma 5.10. *Let \mathcal{C}_0 be as in Lemma 5.5. Then there exists C_0 such that every leaf of \mathcal{C}_0 is a C -quasigeodesic in \widetilde{M} .*

Proof. Suppose not. Then, by Lemma 5.9, there exists an end E such that $\mathcal{L}_E \subset \mathcal{C}_0$, where \mathcal{L}_E is the ending lamination for an end E with $\partial E = S$.

Let Θ be a complementary ideal polygon of \mathcal{L}_E in \widetilde{S} . Let $\gamma_i : i = 1, 2, \dots, k$ be the (infinite) sides of Θ . Note that by Theorem 2.2, all the points $\partial i(\gamma_i(\pm\infty))$ are the same. Let z denote this point on $\partial\widetilde{M}$. Let P_i be the totally geodesic plane on which γ_i lies. There exist $x_i \in \gamma_i$ such that the diameter of the set $\{x_1, \dots, x_k\}$ is bounded by $k\delta$ (where δ is the hyperbolicity constant of \widetilde{S}). Hence the geodesic rays $[x_i, z]$ lying on P_i and asymptotic to z all lie in $k\delta$ neighborhoods of each other. Next for any two successive sides γ_i, γ_{i+1} there are infinite rays contained in each (say the forward directed ray $\gamma_{i,+}$ in γ_i and the backward directed ray $\gamma_{i+1,-}$ in γ_{i+1}) which lie at bounded distance from each other. Hence the convex hull $H_{i,+}$ of $[x_i, z] \cup \gamma_{i,+}$ (which lies in P_i) and the convex hull $H_{i+1,-}$ of $[x_{i+1}, z] \cup \gamma_{i+1,-}$ (which lies in P_{i+1}) also lie at bounded distance from each other.

Translate all the $H_{i,+}, H_{i,-}$'s by a sequence of elements g_n of $G (= PSL_2(\mathbb{C}))$ that translate along $[x_1, z]$ by pulling a point $p_n \in [x_1, z]$ with $d(p_n, x_1) = n$ back to x_1 . Then, since $H_{i,+} \cup H_{i,-} \subset P_i$, we obtain in the limit a family of totally geodesic planes $P_{i,\infty}$, one for each i . Hence we have, in the limit, k round circles $\partial P_{i,\infty}$ on $\partial\widetilde{M}$. (Note here that the elements g_n do not lie in Γ , but rather in $G = PSL_2(\mathbb{C})$.)

On the other hand, by the structure of Cannon-Thurston maps (Theorem 2.2), the limit of $\cup_i (H_{i,+} \cup H_{i,-})$ has as its boundary the one point compactification of the k -pronged singularity corresponding to Θ , or equivalently a space $\overline{\Theta}$ homeomorphic to the suspension of k points. Since $\overline{\Theta}$ cannot be homeomorphic to the union of k round circles, we have a contradiction. \square

Remark 5.11. *We note here that the proof of Lemma 5.10 above **does not** use minimality of X . We first give the analogous statement when X is an H -invariant X generated by a single plane P , i.e. $\widetilde{X} = \overline{\Gamma P}$. (This is adequate for the removal of inessential and asymptotically inessential loops.) We use the X thus obtained to define \mathcal{C} as in Lemma 5.5. The proof of Lemma 5.10 now goes through to establish that leaves of \mathcal{C} are uniform quasigeodesics in \widetilde{M} .*

Reverse engineering the argument now, suppose that a leaf l of the ending lamination \mathcal{L}_E is in \mathcal{C} for some H -invariant X_0 properly contained in M . Then $l \subset \widetilde{P} \cap \text{bdy}(\widetilde{K})$ for some $P \in X_0$. Let X be the H -invariant set generated by P . Applying the argument in the previous paragraph shows that leaves of \mathcal{C} are uniform quasigeodesics in \widetilde{M} . In particular, no leaf of \mathcal{L}_E is in \mathcal{C} - a contradiction. We thus conclude that the conclusions of Lemma 5.10 remain valid for any closed H -invariant X properly contained in M .

6. THE MAIN THEOREM

We are now in a position to define U -minimal sets contained in the H -minimal set X . Let \mathcal{C}_0 be the algebraic lamination furnished by Lemma 5.6. Lemma 5.10 then guarantees that leaves of \mathcal{C}_0 are uniform quasigeodesics in \widetilde{M} . Remark 5.8 then guarantees that a sublamination \mathcal{C} of \mathcal{C}_0 is carried by a boundary component $S \subset \text{bdy}(K)$. We replace each leaf l' of \mathcal{C} by the unique bi-infinite geodesic l in \widetilde{M} that tracks it. Let Y_0 be the set of all (bi-infinite) horocycles σ given by the following:

For each triple $\{(P, l, p) : P \in X, l \in \mathcal{C}, p \in l\}$ there are precisely two horocycles $\sigma_{\pm}(P, l, p)$ contained in \widetilde{P} passing through p and converging at infinity to $l(\pm\infty)$. The union of all such horocycles is denoted by Y .

By Proposition 4.1, none of these horocycles are properly embedded in M . Since the topology on horocycles consists of the topology of circles in B^3 tangential to S^2 minus singletons, Y_0 is closed (since \mathcal{C} is closed as is X) and equivariant under $\pi_1(S)$.

The same argument as in Proposition 3.3 now gives:

Proposition 6.1. *There exists a minimal closed U -invariant subset $Y \subset Y_0$. Further, for all $y \in Y$, $Y = \overline{Uy}$.*

Proof. Each horocycle σ meets a bi-infinite geodesic $l(\in \mathcal{C})$ at right angles in some $P \in X$. Here l lies at a uniformly bounded distance from \widetilde{S} . Hence there exists a compact set $Q \subset M$ such that the image of σ in M passes through Q infinitely often. We partially order closed U -invariant subsets U_{α} of Y_0 as follows: $U_{\alpha} < U_{\beta}$ if $U_{\alpha} \cap Q \subset U_{\beta} \cap Q$. Since Q is compact, any totally ordered chain has a lower bound (by taking intersections). Hence, by Zorn's Lemma, there exists a minimal closed U -invariant set. \square

6.1. Reduction to two cases. We now state the main theorem of the paper. The rest of the Section is devoted to its proof.

Theorem 6.2. *Let Γ be a degenerate Kleinian group without parabolics and $M = \mathbf{H}^3/\Gamma$ be the associated degenerate hyperbolic 3-manifold. Suppose further that M has an incompressible core. Let X be a minimal closed H -invariant subset, where $H = \text{PSL}_2(\mathbb{R})$. Then X is either an immersed totally geodesic surface or all of M .*

After obtaining closed minimal U -invariant sets Y from Proposition 6.1, we shall follow the overall plan of [Mar89, MMO15a]. We thus obtain a sequence $g_n \in (G \setminus H)$ such that $g_n \rightarrow 1$ and $g_n Y = Y$. From this, using Theorem 2.13, we extract a sequence $v_n \in AV$ such that $v_n \rightarrow 1$ and $v_n Y = Y$.

First since Y is closed minimal and U -invariant, it follows that for every $y \in Y$, $\overline{Uy} = Y$. Further, Proposition 4.1 guarantees that $Y \neq Uy$, i.e. the U -minimal set Y **does not** consist of single orbit. We now want to find a sequence of small elements $g_n \in (G \setminus H)$ such that $g_n Y = Y$. By minimality, it suffices to find g_n such that $g_n Y \cap Y \neq \emptyset$.

Fix a horocycle σ in Y . Then by minimality of Y , $\overline{\sigma} = Y$. Let P be the immersed totally geodesic plane containing the horocycle σ in Y . Then there exists a sequence $\{z_n\}$ satisfying the following.

- (1) $z_n \in Uy$.
- (2) $\{z_n\}$ is Cauchy in M .

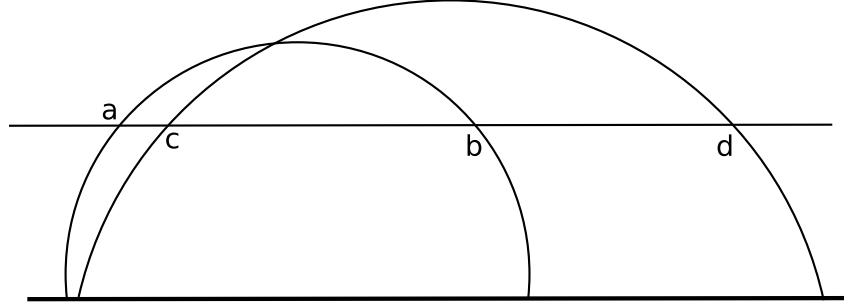
- (3) There is a sequence of geodesics $[z_n, z_m]$ joining z_n, z_m in M such that the lengths $l([z_n, z_m])$ tend to zero.

By choosing the length of the horocycle segment between z_n, z_m large enough, we can assume that the unique geodesic (z_n, z_m) lying on P , joining z_n to z_m , and path-homotopic to the corresponding horocycle segment (contained in Uy) is almost perpendicular to Uy at its end-points. Hence the two ends of (z_n, z_m) are nearly parallel. Joining them by $[z_n, z_m]$ gives a closed loop γ_{mn} arbitrarily close to a closed geodesic. Two possibilities arise:

Case A: The geodesic realizations of γ_{mn} lie on P for infinitely many pairs m, n .

Case B: The geodesic realizations of γ_{mn} do not lie on P for all but finitely many pairs m, n .

6.2. Proof of Main Theorem in Case A. We lift the points z_n to a totally geodesic plane \tilde{P} in \tilde{M} . Also let $\tilde{\sigma}$ be the lift of the horocycle on which the points $\{z_n\}$ lie. Then there exist (at least) four points a, b, c, d on $\tilde{\sigma}$ such that the geodesics $(a, b), (c, d)$ lie very close to lifts of the closed geodesics $\gamma_1, \gamma_2 \in \{\gamma_{mn}\}$. Extend $(a, b), (c, d)$ infinitely in both directions to get bi-infinite geodesics $(a_\infty, b_\infty), (c_\infty, d_\infty)$. (See Diagram below.)



Since the collection γ_{mn} is infinite, we can choose a, b, c, d and a base-point $o \in \tilde{P}$ in such a way that

- (1) the visual angles between the pairs $(a_\infty, \gamma_1(-\infty)), (b_\infty, \gamma_1(+\infty)), (c_\infty, \gamma_2(-\infty)), (d_\infty, \gamma_2(+\infty))$ are all small.
- (2) the distance between b, d is large.

It follows that at least three of points $\gamma_1(-\infty), \gamma_1(+\infty), \gamma_2(-\infty), \gamma_2(+\infty)$ are distinct points on the boundary S^2 . Since Γ is discrete without parabolics, it follows that all four points $\gamma_1(-\infty), \gamma_1(+\infty), \gamma_2(-\infty), \gamma_2(+\infty)$ are distinct and hence the group generated by γ_1, γ_2 is non-elementary. In other words, the stabilizer of P is non-elementary. If P is already a closed immersed surface in M , then there is nothing left to prove for Theorem 6.2. Else, choose $P_1 \neq P$ such that P_1 is an immersed plane in the minimal set X . Then $P \subset \overline{P_1}$ and the hypotheses for Theorem 2.11 are satisfied. It follows that $\Gamma(\partial \tilde{P}_1) = \mathcal{S}$; or equivalently $X = M$.

6.3. Proof of Main Theorem in Case B. Case B implies that (for infinitely many pairs z_n, z_m) the short geodesic segments $[z_n, z_m]$ necessarily join points z_n, z_m that lie in different sheets of $B \cap P$ for some small ball B in M . Hence there exist small elements $g_j \in G \setminus H$ such that $g_j Y \cap Y \neq \emptyset$ (e.g. choose g_{mn} such that $g_{mn} z_m = z_n$). Since $\overline{Uy} = Y$ for all $y \in Y$, this immediately gives (cf. [MMO15a, Lemma 9.5])

Lemma 6.3. *For any $y \in Y$ there exists a sequence $g_n \rightarrow 1$ in $G \setminus H$ such that $g_n y \in Y$.*

By Lemma 6.3, there is a sequence $g_n \rightarrow id$ in $G \setminus H$ such that $g_n Y \cap Y \neq \emptyset$. Since $UY = Y$ is minimal, it follows that $g_n Y = Y$. From g_n we extract a sequence $f_n \in AV \setminus \{1\}$ such that $f_n \rightarrow 1$; $f_n \neq 1$ and $f_n Y = Y$.

If $g_n \in AN$ for infinitely many n , then since $g_n UY = g_n Y = Y$, we can post-multiply g_n by the inverse of its U component to get a sequence $f_n \rightarrow 1$ with $f_n \in AV \setminus \{1\}$ satisfying $f_n Y = Y$.

Else let $g_n \notin AN$ for all but finitely many n . Since $Ug_n UY = Y$, it follows from Theorem 2.13 that for any neighborhood G_0 of $1 \in G$, there exist $u_n, u'_n \in U$ such that $u_n g_n u'_n \rightarrow f \in G_0 \cap (AV \setminus id)$. Since $Ug_n UY = Y$, and Y is closed, we have $fY = Y$. Further, since G_0 is arbitrary, there exists a sequence $f_m \rightarrow 1$ in $(AV \setminus id)$ satisfying $f_m Y = Y$.

Thus, in either case, there exists a sequence $f_m \rightarrow 1$ in $(AV \setminus id)$ such that every element of the cyclic group $\langle f_m \rangle$ preserves Y , i.e. $\langle f_m \rangle Y = Y$. Passing to a subsequence and taking a Hausdorff limit, we obtain a closed, 1-parameter group $L \subset AV$ such that $LY = Y$. Since X is H -minimal it follows that $LX = X$ since for every $\lambda \in L$ $Y (= \lambda Y) \subset \lambda X \cap X$ forcing $\lambda X = X$.

The cosets $LHy \subset X$ give a non-constant, continuous family of circles whose union contains an open subset of S^2 . Hence, by Theorem 2.10 the collection of circles $\{\partial P : P \in X\}$ is all of \mathcal{S} . Hence $X = M$. This completes the proof of Theorem 6.2. \square

6.4. Remarks on the compressible core case. The proof of Theorem 6.2 deals with Alternative A of Proposition 5.4, where a boundary component S of the compact core K carries an algebraic lamination essentially given by $X \cap K$. As Lemma 5.5 points out, incompressibility of K is sufficient to guarantee this alternative. It seems to us that Alternative B will require quite different techniques to handle. The test-case is when Γ is a degenerate handlebody group, i.e. it is free without parabolics and has S^2 as its limit set.

7. COMPACT TOTALLY GEODESIC SURFACES

For the purposes of this Section, we relax the assumption that Γ is degenerate and that M has incompressible core. However, we do assume that Γ has no parabolics and that it has a compact core K . In other words the main Theorem of this section applies to finitely generated Kleinian groups without parabolics. Let Λ_Γ denote the limit set of Γ . We first recall some material from [LM16] to which we refer for details.

7.1. Almost minimizing geodesics.

Definition 7.1. *A geodesic $\gamma = \gamma(t) : t \in \mathbb{R}$ in M is called **almost minimizing** if it has unit speed and if there exist $C \geq 0$ such that $d_M(\gamma(0), \gamma(s)) \geq |s| - C$.*

It is easy to construct almost minimizing geodesic rays. Choose a sequence z_n exiting an end E , join them to $S (= \partial E)$ by minimizing geodesic segments, and take a limit. Any limiting ray is an almost minimizing geodesic ray. It follows from work of Ledrappier [Led97, Proposition 4] and Eberlein [Ebe73, Proposition 5.6] that the set of landing points in S_∞^2 of (lifts to \widetilde{M} of) almost minimizing geodesic rays coincides with the complement of the *horospherical limit set* $\Lambda_h(\Gamma)$ of Γ . We

shall not need this in what follows; however we shall denote the collection of landing points of almost minimizing geodesic rays by $\Lambda_h(\Gamma)^c$.

Let ∂i denote the (boundary value of) the Cannon-Thurston map from the Gromov boundary $\partial\Gamma$ to its limit set Λ_Γ .

Definition 7.2. *The multiple limit set* $\Lambda_m(\Gamma) = \{x \in \Lambda_\Gamma : \#|(\partial i)^{-1}(x)| > 1\}$.

Equivalently, by Theorem 2.2,

$$\Lambda_m = \{\partial i(y) : y \text{ is an end-point of a leaf of } \mathcal{L}_E \text{ for some ending lamination } \mathcal{L}_E\}.$$

In [LM16, Section 3], we establish

Proposition 7.3. [LM16] $\Lambda_h(\Gamma)^c = \Lambda_m$.

We include a proof-sketch for completeness:

Fix a degenerate end E of M . To show that $\Lambda_H^c \subset \Lambda_m$ it suffices to show that if \tilde{r} is the lift of an almost minimizing geodesic r to \tilde{E} , then it lands on Λ_m . Choose a sequence of surfaces S_i exiting the end E . Choose homotopically essential simple closed curves $s_i \subset S_i$ of bounded length (bounded by C_0 say) such that $r \cap s_i \neq \emptyset$ for all i . Choose simple closed curves a_i on $S_0 = \partial E$ such that $r(0) \in a_i$ for all i (we are just freely homotoping s_i down to S_0 making sure that the curve passes through $r(0)$). Consider the annulus with boundary components a_i, s_i and containing the segment r_i of r from $r(0)$ to $r \cap s_i$. Lifting to the universal cover we obtain a quadrilateral whose ‘top edge’ (corresponding to s_i) has length bounded by C_0 and whose ‘bottom edge’ (corresponding to a_i) has length tending to infinity as $i \rightarrow \infty$. Let \tilde{r}_i (resp. \tilde{r}_i^o) be the lift of r_i joining the beginning points of a_i, s_i (resp. the lift of r_i with orientation reversed joining the end points of s_i, a_i). It follows that

- (1) The lifts of curves a_i to \tilde{M} converge to a leaf l of the ending lamination for E .
- (2) The Cannon-Thurston map identifies $l(-\infty), l(\infty)$ to $\tilde{r}(\infty)$.

Thus $\tilde{r}(\infty) \in \Lambda_m$.

Conversely, if $z \in \Lambda_m$, then by Theorem 2.2, there is an end E and a bi-infinite leaf (a, b) of the ending lamination \mathcal{L}_E corresponding to the end E such that $\hat{i}(a) = \hat{i}(b) = z$. Then [Thu80, Chapter 9] there exists a sequence a_n of simple closed geodesics on $S(= \partial E)$ such that

- (1) $a_n^{\pm\infty}$ converges to $\{a, b\}$ (where $a_n^{\pm\infty}$ denote the attracting and repelling fixed points of a_n acting on ∂G , the Gromov boundary of G).
- (2) The geodesic realizations s_n of a_n in E exit E .

Joining s_n to a_n by a geodesic r_n that realizes $d_M(s_n, a_n)$ and taking limits as $n \rightarrow \infty$, we see that r_n converges (up to subsequencing) to an almost minimizing geodesic ray r in E , such that $r(\infty) = \hat{i}(a) = \hat{i}(b)$. This forces $r(\infty) = z$ and hence $z \in \Lambda_H^c$. \square

7.2. Finitely many closed surfaces. The rest of this section is devoted to showing:

Theorem 7.4. *Let Γ be a finitely generated Kleinian group without parabolics and let $M = \mathbb{H}^3/\Gamma$. If M has infinite volume, then there can exist only finitely many compact totally geodesic surfaces in M .*

When Γ is convex cocompact, Theorem 7.4 is a direct consequence of [MMO15a, Corollary B.2]. Therefore we are interested in the case when at least one end E of M is degenerate. We argue by contradiction. Suppose that there is an infinite sequence $\{S_i\}$ of totally geodesic surfaces in M . As an immediate consequence of Theorem 2.12, we have the following:

Lemma 7.5. *Let Γ be a finitely generated Kleinian group without parabolics and let $M = \mathbb{H}^3/\Gamma$. Suppose M has infinite volume. If there is an infinite sequence $\{S_i\}$ of totally geodesic surfaces in M , then there exists an end E such that (after passing to a subsequence if necessary) the sequence $\{S_i\}$ satisfies the following: For any compact subset Q of M , $S_i \cap (E \setminus Q) = \emptyset$ for only finitely many i .*

Proof. Since the fundamental groups $\pi_1(S_i)$ inject into $\pi_1(M)$, the limit sets of $\pi_1(S_i)$ are contained in Λ_Γ , the limit set of Γ . Hence the totally geodesic surfaces $\{S_i\}$ are all contained in the convex core $CC(M)$ of M . By Theorem 2.12, there can be only finitely many of the $\{S_i\}$'s contained in any compact subset of $CC(M)$. Since $CC(M)$ has finitely many ends, the Lemma follows by passing to a subsequence if necessary. \square

In other words, the sequence $\{S_i\}$ penetrates arbitrarily deep into E . Let $S = \text{bdy}(E)$ be the boundary of the end E . It follows, by Lemma 3.1, that we may choose $S = K \cap E$, where K is a compact core of M . Since Γ has no parabolics, it is necessarily a Gromov-hyperbolic group. Also, it follows from Thurston's hyperbolization of atoroidal manifolds with boundary (see [BF92], [Mit04, Theorem 4.6] for a proof in the context of hyperbolic groups) that

Lemma 7.6. *Let $j : S \rightarrow M$ denote the inclusion map. Then $j_*(\pi_1(S)) = \Delta$ is quasiconvex in Γ .*

Let Δ_i denote the subgroup of Γ corresponding to $\pi_1(S_i)$. Since S_i is totally geodesic, Δ_i is quasiconvex in Γ . Since the intersection of quasiconvex groups is quasiconvex [Sho91, Proposition 3], it follows that $\mathcal{G}_i := \Delta \cap \Delta_i$ is quasiconvex in Γ . We assume (after perturbing K slightly if necessary), that S is transverse to S_i for all i . Also each S_i necessarily intersects S (as we have chosen the sequence this way).

Let M_i be the cover of M corresponding to the subgroup \mathcal{G}_i . Let $\overline{S_i}$ and Σ_i denote the unique lifts of S_i, S respectively to M_i , such that $\pi_1(\overline{S_i}) = \pi_1(\Sigma_i) = \mathcal{G}_i$. Note that $\overline{S_i}$ and Σ_i are embedded submanifolds of M_i , and that M_i is homeomorphic to $\overline{S_i} \times \mathbb{R}$ as well as $\Sigma_i \times \mathbb{R}$. We shall be interested in a compact core of M_i bounded by (pieces of) $\overline{S_i}$ and Σ_i . Thus there exist

- (1) A compact submanifold with boundary $\overline{S_i}^0$ of $\overline{S_i}$ whose inclusion into $\overline{S_i}$ is a homotopy equivalence.
- (2) A compact submanifold with boundary Σ_i^0 of Σ_i whose inclusion into Σ_i is a homotopy equivalence.
- (3) $\overline{S_i}^0 \cap \Sigma_i^0$ is a finite union of circles containing $\text{bdy}(\overline{S_i}^0) = \text{bdy}(\Sigma_i^0)$.

Then there exist compact product regions in M whose boundaries consist of (isotopic) submanifolds of $\overline{S_i}^0$ and Σ_i^0 intersecting only along their boundary circles. We shall consider one of these product regions Q_i . Passing to the cover of M_i corresponding to $\pi_1(Q_i)$ and abusing notation slightly, we set $\mathcal{G}_i = \pi_1(Q_i)$ and assume that the compact manifold Q_i has boundary given by $\text{bdy}(Q_i) = \overline{S_i}^0 \cup \Sigma_i^0$.

Also, the inclusions of \overline{S}_i^0 or Σ_i^0 into Q_i is a homotopy equivalence, as is the inclusion of Q_i into M_i . Thus, by passing to a subsurface if necessary, we are assuming that $\overline{S}_i^0 \cap \Sigma_i^0$ is exactly equal to $\text{bdy}(\overline{S}_i^0) = \text{bdy}(\Sigma_i^0)$; and further that $\overline{S}_i^0 \cup \Sigma_i^0$ bounds the product region Q_i .

Let $\Pi_i : M_i \rightarrow M$ be the covering projection and let E_i be the lift of E containing Σ_i . Since the sequence of totally geodesic surfaces $\{S_i\}$ penetrates arbitrarily deep into E , it follows that there exist points $q_i \in \overline{S}_i^0$ that realize the maximum distance (amongst points in $\overline{S}_i \cap E_i$) from Σ_i . Then after projecting back to M using Π_i , $\Pi_i(q_i)$ is a point in $S_i \cap E$ that maximizes (amongst points in $S_i \cap E$) distance from S .

Let $p_i \in \Sigma_i$ be a point on Σ_i closest to p_i , i.e. $d_i(p_i, q_i) = d_i(\Sigma_i, q_i)$, where d_i denotes the hyperbolic metric on M_i . Then $d_i(p_i, q_i) \rightarrow \infty$ as $i \rightarrow \infty$. Let $[p_i, q_i]$ be the shortest path between p_i, q_i . Then $[p_i, q_i]$ is perpendicular to \overline{S}_i at q_i (since q_i maximizes distance). We note further, that since $[p_i, q_i]$ is a minimizing geodesic segment, it remains a minimizing geodesic segment after projecting back to M . Since a limit of minimizing geodesic segments is an almost minimizing geodesic ray, we have the following.

Lemma 7.7. *Any limit (as $i \rightarrow \infty$) of the sequence $\{\Pi_i([p_i, q_i])\}$ is an almost minimizing geodesic ray.*

We shall also need the following:

Lemma 7.8. *Let \mathcal{A}_i be the family of geodesic rays in \overline{S}_i starting at q_i and let $a_i \in \mathcal{A}_i$ be one of these rays. Then the concatenation $\alpha_i = [p_i, q_i] \cup a_i$ lifted to the universal cover \widetilde{M} is a uniform (independent of $a_i \in \mathcal{A}_i$) quasigeodesic in \widetilde{M} .*

Proof. This follows from the well-known fact that the concatenation of two geodesics perpendicular to each other is a uniform quasigeodesic (see [Mit98, Lemma 3.3] for instance). \square

Consider the family of geodesic segments \mathcal{R}_i (in the intrinsic metric) on Σ_i starting at p_i and ending on $\text{bdy}(\Sigma_i^0) (= \text{bdy}(\overline{S}_i^0))$. Then the concatenation $\gamma_i = [q_i, p_i] \cup r_i$ for $r_i \in \mathcal{R}_i$ is path-homotopic to a geodesic segment contained in a unique $a_i \in \mathcal{A}$. Thus, starting with $r_i \in \mathcal{R}_i$ we perform the following operations:

- (1) First adjoin $[q_i, p_i]$ to its beginning and path-homotop it to a geodesic subsegment of a unique $a_i \in \mathcal{A}_i$.
- (2) Then adjoin $[p_i, q_i]$ to the beginning of the subsegment of a_i thus obtained and get a uniform quasigeodesic segment α_i starting at p_i .
- (3) Note that α_i is path homotopic in M_i to the original $r_i \in \mathcal{R}_i$ we started with.

Next, lift everything to the universal cover \widetilde{M} and assume that all the p_i 's are lifted to lie in a fixed fundamental domain in \widetilde{S} (a lift of S to \widetilde{M}). Let \mathcal{R}_∞ be the family of infinite geodesic rays r_∞ in \widetilde{S} that satisfy the following property: There exists a sequence of geodesic segments $\{r_i \in \mathcal{R}_i\}$ such that $r_i \subset r_\infty$ for all i and $\cup_i r_i = r_\infty$.

Let $\partial\mathcal{R}_\infty$ denote the collection of landing points of these rays in $\partial\Gamma$ (the Gromov boundary of Γ). Recall that $j_*(\pi_1(S)) = \Delta$ is quasiconvex in Γ by Lemma 7.6. Clearly, $\partial\mathcal{R}_\infty \subset \partial\Delta$. Also if the rays \mathcal{R}_i are extended infinitely they would land on $\partial\Delta \setminus \partial\mathcal{G}_i$. Hence $\partial\Delta \setminus (\cup_i \partial\mathcal{G}_i) \subset \partial\mathcal{R}_\infty$. Note that since \mathcal{G}_i is quasiconvex in Δ ,

it follows that $\partial\mathcal{G}_i$ is nowhere dense in $\partial\Delta$ and also has zero measure (with respect to any visual measure). We have shown

Lemma 7.9. *$\partial\Delta \setminus \partial\mathcal{R}_\infty$ is nowhere dense in $\partial\Delta$ and has zero measure with respect to any visual measure on it.*

Recall that any limit (as $i \rightarrow \infty$) of $\Pi_i([p_i, q_i])$'s is almost minimizing by Lemma 7.7. Since $\alpha_i = [p_i, q_i] \cup a_i$ is a uniform quasigeodesic by Lemma 7.8, it follows that all limits of the α_i 's (as $i \rightarrow \infty$) are also almost minimizing. Let \mathcal{B} denote the set of limiting rays of the $\{\alpha_i\}$'s. Let $\partial\mathcal{B}$ denote the set of landing points in S_∞^2 of rays in \mathcal{B} .

Note now that each ray $r_\infty \in \mathcal{R}_\infty \subset \tilde{S}$ is a limit of $r_i \in \mathcal{R}_i$. Let $r_\infty(\infty)$ denote its landing point in the Gromov boundary $\partial\Delta \subset \partial\Gamma$. We have seen above that there exists a uniform quasigeodesic α_i in M_i path-homotopic to r_i . Since a Cannon-Thurston map ∂i exists by Theorem 2.2, it follows that $\partial i(r_\infty(\infty))$ belongs to $\partial\mathcal{B}$ for all $r_\infty \in \mathcal{R}_\infty$. By Proposition 7.3, it follows that $\partial i(r_\infty(\infty)) \in \Lambda_m$, the multiple limit set, for all $r_\infty \in \mathcal{R}_\infty$. Hence by Theorem 2.2, $r_\infty(\infty) \in \partial\Delta$ is an ideal end-point of a leaf of the ending lamination \mathcal{L}_E corresponding to the end E . The set of all such end-points is of zero measure and is nowhere dense (see [Thu80, Ch. 8] and [Som95] for instance). This contradicts Lemma 7.9 and proves Theorem 7.4. \square

ACKNOWLEDGMENTS

I am extremely grateful to Anish Ghosh for several exciting and instructive conversations. Special thanks are due to Curt McMullen for raising the question [McM16] that Theorem 7.4 answers; and also for insightful comments on the Ratner-type phenomenon explicated in [MMO15a].

REFERENCES

- [BCM12] J. F. Brock, R. D. Canary, and Y. N. Minsky. The Classification of Kleinian surface groups II: The Ending Lamination Conjecture. *Ann. of Math.* 176 (1), *arXiv:math/0412006*, pages 1–149, 2012.
- [BF92] M. Bestvina and M. Feighn. A Combination theorem for Negatively Curved Groups. *J. Diff. Geom.*, vol 35, pages 85–101, 1992.
- [BFH97] M. Bestvina, M. Feighn, and M. Handel. Laminations, trees and irreducible automorphisms of free groups. *Geom. Funct. Anal.* vol.7 No. 2, pages 215–244, 1997.
- [CHL08] T. Coulbois, A. Hilion, and M. Lustig. R-trees and laminations for free groups I: algebraic laminations. *J. Lond. Math. Soc.* (2), 78(3), pages 723–736, 2008.
- [DM16] S. Das and M. Mj. Semiconjugacies Between Relatively Hyperbolic Boundaries. *to appear in Groups, Geometry and Dynamics*, *arXiv:1007.2547*, 2016.
- [Ebe73] P. Eberlein. Geodesic flows on negatively curved manifolds, ii. *Trans. of the A.M.S.*, Vol. 178, pages 57–82, 1973.
- [Hed36] G. A. Hedlund. Fuchsian groups and transitive horocycles. *Duke Math. J.*, 2:530–542, 1936.
- [KL10] I. Kapovich and M. Lustig. Intersection form, laminations and currents on free groups. *Geom. Funct. Anal.* 19, no. 5, pages 1426–1467, 2010.
- [Led97] F. Ledrappier. Horospheres on abelian covers. *Bol. Soc. Brasil. Mat. (N.S.)* 28 no. 2, pages 363–375, 1997.
- [LM16] C. Lecuire and M. Mj. Horospheres in degenerate 3-manifolds. *preprint*, 2016.
- [Mar89] G. A. Margulis. Indefinite quadratic forms and unipotent flows on homogeneous spaces. *In Dynamical systems and ergodic theory (Warsaw, 1986)*, Banach Center Publ., 23, 1989.
- [McM16] C. McMullen. personal communication. 2016.

- [Min10] Y. N. Minsky. The Classification of Kleinian surface groups I: Models and Bounds. *Ann. of Math.* 171(1), *math.GT/0302208*, pages 1–107, 2010.
- [Mit97] M. Mitra. Ending Laminations for Hyperbolic Group Extensions. *Gem. Funct. Anal.* vol.7 No. 2, pages 379–402, 1997.
- [Mit98] M. Mitra. Cannon-Thurston Maps for Trees of Hyperbolic Metric Spaces. *J. Differential Geom.* 48, pages 135–164, 1998.
- [Mit99] M. Mitra. On a theorem of Scott and Swarup. *Proc. A. M. S.* v. 127 no. 6, pages 1625–1631, 1999.
- [Mit04] M. Mitra. Height in Splittings of Hyperbolic Groups. *Proc. Indian Acad. of Sciences*, v. 114, no.1, pages 39–54, Feb. 2004.
- [Mj10] M. Mj. Cannon-Thurston Maps for Kleinian Groups. *preprint*, *arXiv:math 1002.0996*, 2010.
- [Mj14a] M. Mj. Cannon-Thurston Maps for Surface Groups. *Ann. of Math.*, 179(1), pages 1–80, 2014.
- [Mj14b] M. Mj. Ending Laminations and Cannon-Thurston Maps, with an appendix by S. Das and M. Mj. *Geom. Funct. Anal.* 24, pages 297–321, 2014.
- [MMO15a] C. T. McMullen, A. Mohammadi, and H. Oh. Geodesic planes in hyperbolic 3-manifolds. *preprint*, 2015.
- [MMO15b] C. T. McMullen, A. Mohammadi, and H. Oh. Horocycles in hyperbolic 3-manifolds. *preprint*, 2015.
- [MR15] M. Mj and K. Rafi. Algebraic Ending Laminations and Quasiconvexity. *preprint*, *arXiv:1506.08036*, 2015.
- [Rat91a] M. Ratner. On Raghunathans measure conjecture. *Ann. of Math.* 134, pages 545–607, 1991.
- [Rat91b] M. Ratner. Raghunathans topological conjecture and distributions of unipotent flows. *Duke Math. J.* 63, pages 235 – 280, 1991.
- [Sha91] N. A. Shah. Closures of totally geodesic immersions in manifolds of constant negative curvature. *Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, A. Verjovsky eds.)*, World Scientific, 1991.
- [Sho91] H. Short. Quasiconvexity and a theorem of Howson’s. *Group Theory from a Geometrical Viewpoint (E. Ghys, A. Haefliger, A. Verjovsky eds.)*, 1991.
- [Som95] T. Soma. Equivariant, almost homeomorphic maps between S^1 and S^2 . *Proc. Amer. Math. Soc.* 123, no. 9, pages 2915–2920, 1995.
- [Thu80] W. P. Thurston. The Geometry and Topology of 3-Manifolds. *Princeton University Notes*, 1980.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH. 5, HOMI BHABHA ROAD, MUMBAI-400005, INDIA

E-mail address: mahan@math.tifr.res.in